2023-24 MATH2048: Honours Linear Algebra II

Homework 3

Due: 2023-09-29 (Friday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

   Proof. Linear transformations in vector spaces are determined by their action on a basis. Consider the vectors $(1,1)$ and $(2,3)$. They are linearly independent because no scalar multiple of one is equal to the other. Then $\{(1,1),(2,3)\}$ is a basis for $\mathbb{R}^2$. Now, we can define a linear transformation T by its action on these vectors.

   We have $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. Note that $(8,11) = 2(1,1) + 3(2,3)$, so $T(8,11) = 2T(1,1) + 3T(2,3) = 2(1,0,2) + 3(1,-1,4) = (5,-3,16)$. □

2. Let V be a finite-dimensional vector space and $T : V \to V$ be linear.

   (a) Suppose that $V = R(T) + N(T)$. Prove that $V = R(T) \oplus N(T)$.

   (b) Suppose that $R(T) \cap N(T) = \{0\}$. Prove that $V = R(T) \oplus N(T)$.

   (c) Give an example of V and T such that $V = R(T) \oplus N(T)$.

   Be careful to say in part (a)(b) where finite-dimensionality is used.

   Proof. Because $\dim(V)$ is finite, by Rank-Nullity Theorem, $\dim(V) = \dim(R(T)) + \dim(N(T)) = \dim(R(T) + N(T)) - \dim(R(T) \cap N(T))$. Then

$$
\begin{aligned}
V = R(T) + N(T) &\iff \dim(V) = \dim(R(T) + N(T)), \\
&\iff \dim(R(T) \cap N(T)) = 0, \\
&\iff R(T) \cap N(T) = 0.
\end{aligned}
$$

Hence,

$$V = R(T) + N(T) \iff R(T) \cap N(T) = 0 \iff V = R(T) \oplus N(T).$$

This proves (a) and (b).

(c) Consider the vector space $V = \mathbb{R}^2$ and the linear transformation $T : V \to V$ defined by $T(x, y) = (x, 0)$.

In this case, the range of T is $R(T) = \{(x, 0) : x \in \mathbb{R}\}$ and the null space is $N(T) = \{(0, y) : y \in \mathbb{R}\}$. It's easy to see that $V = \mathbb{R}^2 = R(T) \oplus N(T)$. □

3. Let V be an n-dimensional vector space with an ordered basis β. Define $T : V \to F^n$ by $T(x) = [x]_\beta$. Prove that T is linear.

Proof. Denote the ordered basis β of V as $\beta = \{v_1, v_2, \ldots, v_n\}$.

For any two vectors $u, v \in V$, we can write them as $u = \sum_{i=1}^n a_i v_i$ and $v = \sum_{i=1}^n b_i v_i$ for some scalars $a_i, b_i \in F$. For any scalar $c \in F$, the sum $u + cv = \sum_{i=1}^n (a_i + cb_i) v_i$.

By the definition of T, we have $T(u) = [u]_\beta = (a_1, a_2, ..., a_n)^T$ and $T(v) = [v]_\beta = (b_1, b_2, ..., b_n)^T$.

Then $T(u) + cT(v) = (a_1 + cb_1, a_2 + cb_2, ..., a_n + cb_n)^T = [u + cv]_\beta = T(u + cv)$.

Therefore, T is a linear transformation. □

4. Let V be a vector space with the ordered basis $\beta = \{v_1, v_2, ..., v_n\}$. Define $v_0 = 0$. By Theorem 2.6 (p. 72), there exists a linear transformation $T : V \to V$ such that $T(v_j) = v_j + v_{j-1}$ for $j = 1, 2, ..., n$. Compute $[T]_\beta$.

Proof. The image of each vector v_j under T is $T(v_j) = v_j + v_{j-1}$, which translates to two 1's in the j-th and $(j-1)$-th positions in the column vector $[T(v_j)]\beta$, except for $j = 1$ where $[T(v_1)]\beta = (1, 0, ..., 0)^T$.

Therefore, the matrix representation $[T]_\beta$ is:

$$[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

□

5. Let V and W be vector spaces such that $\dim(V) = \dim(W)$, and let $T : V \to W$ be linear. Show that there exist ordered bases β and γ for V and W, respectively, such that $[T]_\beta^\gamma$ is a diagonal matrix.

   Proof. Let n denote the common dimension of V and W. Let $\{v_1, v_2, \ldots, v_k\}$ be a basis for the null space $N(T)$ of T, and extend this basis to $\beta = \{v_1, \ldots, v_n\}$, a basis for V.

   As per the Rank-Nullity theorem, we know that the vectors $\{T(v_{k+1}), \ldots, T(v_n)\}$ form a basis for the range $R(T)$ of T. We can extend this basis to form a basis $\gamma = \{w_1, \ldots, w_k, T(v_{k+1}), \ldots, T(v_n)\}$ for W.

   Now, for $1 \le i \le k$, the image $[T(v_i)]_\gamma$ is the zero vector in F^n, since $v_i \in N(T)$. For $k < i \le n$, however, $T(v_i)$ is a basis vector in $R(T)$, and therefore $[T(v_i)]_\gamma$ is the column vector with a single 1 at the i-th row and 0's elsewhere.

   Therefore, the matrix representation $[T]_{\gamma\beta}$ is a diagonal matrix with 1's corresponding to the vectors in $R(T)$ and 0's corresponding to the vectors in $N(T)$:

$$[T]_{\gamma\beta} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 1 \end{pmatrix}$$

□